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Towards supergravity duals of chiral symmetry breaking in Sasaki-Einstein cascading quiver theories

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Abstract: We construct a first order deformation of the complex structure of the cone over Sasaki-Einstein spaces $Y^{p,q}$ and check supersymmetry explicitly. This space is a central element in the holographic dual of chiral symmetry breaking for a large class of cascading quiver theories. We discuss a solution describing a stack of N D3 branes and M fractional D3 branes at the tip of the deformed spaces.

Keywords: AdS-CFT and dS-CFT Correspondence.

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1. Introduction and summary

One of the most promising elements of the AdS/CFT correspondence is the possibility of providing an alternative method of studying aspects of confining theories. Unfortunately, there are but a few smooth supergravity backgrounds potentially dual to confining $\mathcal{N} = 1$ SYM.

Recently, a new class of Sasaki-Einstein manifolds $Y^{p,q}$ has been constructed [1, 2]. Given a Sasaki-Einstein manifold X^5 one can consider a stack of N D3 branes at the tip of the cone over X^5. Taking the Maldacena limit leads to a duality between string theory on $AdS_5 \times X^5$ and a superconformal gauge theory living in the world volume of the D3 branes. In this context the infinite family of spaces $Y^{p,q}$ was shown to be dual to superconformal quiver gauge theories [3, 4]. Remarkably, the correspondence has provided a better treatment of some of the gauge theory questions. In particular, the irrational nature of some R-charges has been elucidated in the field theory using gravity input [3, 5].

Since $Y^{p,q}$ spaces are generalizations of $T^{1,1}$ one naturally wonders about the possibility of further generalizing these spaces in a way that parallels the program carried by Klebanov and collaborators for the conifold [6, 7]. In fact, the first step in this direction has already been taken in a recent collaboration including Klebanov [8]. The pinnacle of Klebanov's program is the Klebanov-Strassler (KS) background [7], which is a smooth supergravity solution constructed as a warped deformed conifold.

In this paper we take a step in the construction of generalizations of the KS background. Namely, we construct a first order deformation of the complex structure of the cone over $Y^{p,q}$. This deformation should lead, upon warping, to a supergravity background describing the IR of the recently constructed cascading solution [8] on the cone over $Y^{p,q}$.

One important motivation for this work is that the existence of a supergravity background allows for explicit computation of some dynamical quantities, like the spectrum of various states including mesons, quantum corrections to Regge trajectories and possibly Goldstone bosons.

In section 2 we review some of the key aspects of the $Y^{p,q}$ spaces and their dual quiver gauge theories. Our aim is to set up the notation and technical motivation for later sections. In particular, we address various technical aspects as the supergravity realization of chiral symmetry breaking and the algebra of vielbeins. In section 3, after recalling the general theory of deformation of CY spaces from the differential geometric point of view, we present the first order deformation for the cone over $Y^{p,q}$. This is the main technical result of the paper. Section 3 also contains the result of explicitly checking that the deformation is supersymmetric. This guarantees that the deformation has SU(3) holonomy and that indeed corresponds to a complex Kähler manifold. Section 4 reviews the complex coordinates introduced for the cone over $Y^{p,q}$ in [3] with the aim of classifying the deformation. We show that it corresponds to a deformation of the complex structure. Moreover, in this section we find the Kähler potential for the cone over $Y^{p,q}$, knowing the Kähler potential opens a new venue for understanding deformations of these CY spaces. Section 5 discusses placing a stack of N D3 and M fractional D3 branes at the tip of the deformed space. Since we only know the deformation of the space to first order, our analysis is approximate. The main technical result in this section is the explicit construction of the imaginary self dual 3-form in the deformed space. This implies that to first order there is a supergravity background that should correspond to the chiral symmetry broken phase of the cascading Sasaki-Einstein quivers discussed in [8].

Let us end this section with some comments on open questions that we were not able to answer in this work. The most glaring question is the existence of a solution beyond first order. Our paper follows a strictly differential geometric approach to the deformation. Another unexplored venue is the algebraic approach. Namely, the complex deformation for the conifold is fairly simple from the algebraic standpoint. Given the defining equation $\sum_{a=1}^{4} w_a^2 = 0$, we simply need to replace it by $\sum_{a=1}^{4} w_a^2 = \epsilon^2$. There are various questions that the algebraic approach answers immediately. For example, chiral symmetry breaking is nothing but the breaking of the $U(1)$ symmetry that rotates the coordinates w_a in the undeformed space. In this paper we followed exclusively the differential geometric approach but we hope to discuss the algebraic one in the future. Finally, chiral symmetry breaking can also be explored along the lines of [9, 10].

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Note Added

After the first version of this work appeared in the archive, a series of papers addressing the complex deformation from the field theory side have given arguments in favor of supersymmetry breaking [11]. We also became aware of work by Altmann [12] who shows the existence of an obtruction for finding the complex deformations beyond first order. It is worth mentioning that while these works make a case for obstruction to a solution built around a complex deformation of the corresponding cone over $Y^{p,q}$ it is plausible that the full solution is supersymmetric albeit with SU(3)-structure rather than SU(3) holonomy as is the case for the baryonic branch of the KS solutions as discussed in [13].

2. Review of superconformal quiver theories and their dual $AdS_5 \times Y^{p,q}$ backgrounds

The gauge theory dual to IIB on $AdS_5 \times Y^{p,q}$ has been the subject of much recent investigation. Here we begin with a summary of some of the key aspects of the quiver gauge theories with $Y^{p,q}$ duals, as explained in ref. [4].

The quivers for $Y^{p,q}$ can be constructed starting with the quiver of $Y^{p,p}$ which is naturally related to the quiver theory obtained from $\mathbb{C}^3/\mathbb{Z}_{2p}$. The gauge group is $\mathrm{SU}(N)^{2p}$ and the superpotential is constructed out of cubic and quartic terms in the four types of fields present: U, V, Y and Z, where U and V are doublets of SU(2):

$$\epsilon_{\alpha\beta} U_a^\alpha V^\beta Y\,, \qquad \epsilon_{\alpha\beta} U_b^\alpha V^\beta Y\,, \qquad \epsilon_{\alpha\beta} Z U_a^\alpha Y U_b^\beta\,. \tag{2.1}$$

Greek indices $\alpha, \beta = 1, 2$ are in SU(2), and Latin subindices a, b refer to the gauge group where the corresponding arrow originates. Equivalently, as explained in [8], the quiver theory for $Y^{p,q}$ can be constructed from two basic cells denoted by σ and τ. Some concrete examples can be found in [4, 8].

One key ingredient is that the geometric realization of the **R**-symmetry is given by a Killing vector field in the Sasaki-Einstein geometry of the form

$$K = \frac{1}{3}\frac{\partial}{\partial\psi}\,, \tag{2.2}$$

where the geometrical meaning of the coordinates ψ and α will become apparent momentarily. Breaking the **R**-symmetry is our main goal, and it translates into constructing a supergravity background without (2.2) as a Killing vector.

2.1 The cone over $Y^{p,q}$

Before turning on any deformations, we highlight the structure of the cone over $Y^{p,q}$, emphasizing its similarities with the conifold, which is the cone over $T^{1,1}$. This connection motivates our later Ansatz for the deformation.

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Following [2, 3], the Sasaki-Einstein metric on $Y^{p,q}$ can be written in the form

$$
\begin{aligned}
ds^2 &= \frac{1-cy}{6}(d\theta^2+\sin^2\theta d\phi^2)+\frac{1}{w(y)v(y)}dy^2+\frac{w(y)v(y)}{36}(d\beta-c\cos\theta d\phi)^2+\\
&\quad+\frac{1}{9}[d\psi+\cos\theta d\phi+y(d\beta-c\cos\theta d\phi)]^2 \qquad (2.3)\\
&= \frac{1}{6}[(e^\theta)^2+(e^\phi)^2+(e^y)^2+(e^\beta)^2]+\frac{1}{9}(e^\psi)^2\,, \qquad (2.4)
\end{aligned}
$$

where

$$
w(y)=\frac{2(a-y^2)}{1-cy}\,,\qquad v(y)=\frac{a-3y^2+2cy^3}{a-y^2}\,. \qquad (2.5)
$$

The natural one forms are given by [8, 15]

$$
\begin{aligned}
e^\theta &= \sqrt{1-cy}\;d\theta\,,\\
e^\phi &= -\sqrt{1-cy}\;\sin\theta d\phi\,,\\
e^y &= -\frac{1}{H(y)}dy,\\
e^\beta &= H(y)(d\beta-c\cos\theta d\phi)\,,\\
e^\psi &= d\psi+\cos\theta d\phi+y(d\beta-c\cos\theta d\phi)\,, \qquad (2.6)
\end{aligned}
$$

with $H(y)=\sqrt{w(y)v(y)/6}$.

Before proceeding, it would be convenient to develop the algebra of exterior derivatives. For this purpose, let $L(y,\theta)=\frac{\cot\theta}{\sqrt{1-cy}}$ and $K(y)=\frac{cH(y)}{2(1-cy)}$. Then $\frac{dH(y)}{dy}=K(y)-\frac{y}{H(y)}$. The exterior derivatives of the above forms then satisfy

$$
\begin{aligned}
de^\theta &= K(y)e^y\wedge e^\theta\,,\\
de^\phi &= K(y)e^y\wedge e^\phi+L(y,\theta)e^\theta\wedge e^\phi\,,\\
de^y &= 0\,,\\
de^\beta &= \left(\frac{y}{H(y)}-K(y)\right)e^y\wedge e^\beta-2K(y)e^\theta\wedge e^\phi\,,\\
de^\psi &= e^\theta\wedge e^\phi-e^y\wedge e^\beta\,. \qquad (2.7)
\end{aligned}
$$

At various stages in the construction, we will refer to the model of the deformed conifold. In particular, it is worth noting that the above definitions reduce to those in [7] describing $T^{1,1}$ in the limit [2]

$$
c\to 0\,,\qquad y\to\cos\theta_2\,,\qquad a\to 3\quad\text{and}\quad \beta\to\phi_2\,. \qquad (2.8)
$$

Following the discussion of [7] (see also [14]), one can further define the rotated and shifted vielbeins $e^1,e^2,\ldots,e^5$ and $g^1,g^2,\ldots,g^5$ as

$$
\begin{aligned}
e^1 &= -e^\beta\,, \qquad e^3=-\cos\psi e^\phi-\sin\psi e^\theta\,,\qquad e^5=e^\psi\,,\\
e^2 &= e^y\,, \qquad e^4=-\sin\psi e^\phi+\cos\psi e^\theta\,, \qquad (2.9)
\end{aligned}
$$

and

$$g^1 = \frac{1}{\sqrt{2}}(e^1 - e^3)\,, \qquad g^3 = \frac{1}{\sqrt{2}}(e^1 + e^3), \qquad g^5 = e^5.$$
$$g^2 = \frac{1}{\sqrt{2}}(e^2 - e^4)\,, \qquad g^4 = \frac{1}{\sqrt{2}}(e^2 + e^4)\,, \tag{2.10}$$

Note that, if desired, one could instead have defined $\tilde{e}^1, \tilde{e}^2, \ldots, \tilde{e}^5$ as

$$\tilde{e}^1 = e^\phi\,, \qquad \tilde{e}^3 = \cos\psi e^\beta - \sin\psi e^y\,, \qquad \tilde{e}^5 = e^\psi\,.$$
$$\tilde{e}^2 = e^\theta\,, \qquad \tilde{e}^4 = \sin\psi e^\beta + \cos\psi e^y\,, \tag{2.11}$$

The corresponding $\tilde{g}^1, \tilde{g}^2, ..., \tilde{g}^5$ would then be defined in a similar fashion as in (2.10) with e^i replaced by $\tilde{e}^i$ for $i = 1, 2, \ldots, 5$. One should note, however, that the $\tilde{e}^i$ are physically indistinct from the e^i in the $T^{1,1}$ limit (2.8). They are distinct when $c \neq 0$, and so we may expect that they could play a rôle in deforming the cone over $Y^{p,q}$. We will nevertheless show that the first order deformation given in terms of the $\tilde{e}^i$ is no different than that given in terms of the e^i.

Although the angular coordinate ψ appears explicitly in the the above one forms, the metric remains ψ independent. As we will see in the next section, the perturbed solution will break this symmetry direction in an analogous way to the solution of [7].

3. The first order complex deformation

We now briefly review the deformation theory of CY spaces following the presentation of [16]. Let the parameter space of Calabi-Yau manifolds be the parameter space of Ricci-flat Kähler metrics, and let g_{mn} and $g_{mn} + \delta g_{mn}$ satisfy

$$R_{mn}(g) = 0, \qquad R_{mn}(g + \delta g) = 0\,. \tag{3.1}$$

Then, with the coordinate condition $\nabla^n \delta g_{mn} = 0$, one obtains that δg_{mn} satisfies the Lichnerowicz equation

$$\nabla^k \nabla_k \delta g_{mn} + 2R_m{}^p{}_n{}^q \delta g_{pq} = 0\,. \tag{3.2}$$

The connection between the geometro-differential approach and the algebraic approach to deforming a CY space arises due to an isomorphism between the solutions of (3.2) and harmonic forms on CY. Namely, a solution with mixed indices is associated with a $(1,1)$-form

$$i\delta g_{\mu\bar{\nu}} dx^\mu \wedge dx^{\bar{\nu}}\,, \tag{3.3}$$

which is harmonic if and only if the variation of the metric satisfies the Lichnerowicz equation, (3.2). Similarly, for a variation of pure type, one can associate a $(2,1)$-form using the holomorphic 3-form Ω:

$$\Omega_{\kappa\lambda}{}^{\bar{\nu}} \delta g_{\bar{\mu}\bar{\nu}} dx^\kappa \wedge dx^\lambda \wedge dx^{\bar{\mu}}\,. \tag{3.4}$$

This form is harmonic if and only if $\delta g_{\bar{\mu}\bar{\nu}}$ satisfies (3.2).

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With these isomorphisms in place we can classify a deformation as either Kähler or complex. In particular, variations of pure type (there are $b_{2,1}$ of them) correspond to variations of the complex structure. Note that $g + \delta g$ is a Kähler metric on a manifold close to the original one. There must therefore exist a coordinate system in which the pure parts of the metric vanish. Under a change of coordinates $x^m \to x^m + f^m(x)$, the metric variation transforms according to the familiar

$$\delta g_{mn} \to \delta g_{mn} - \frac{\partial f^r}{\partial x^m} g_{rn} - \frac{\partial f^r}{\partial x^n} g_{mr} \,. \tag{3.5}$$

If f^μ is holomorphic then $\delta g_{\mu\nu}$ is invariant. Thus, the pure part of the variation could be removed by a transformation of coordinates but it cannot be removed by a holomorphic coordinate transformation. Thus the pure part of the metric variation corresponds precisely to changes of the complex structure.

3.1 First order perturbation

We are now in a position to construct the first order deformation of the cone over $Y^{p,q}$. Using the one-forms of section 2, the undeformed metric is given by

$$ds^2 = dr^2 + r^2 \left(\frac{1}{6} \left[(e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 \right] + \frac{1}{9} (e^5)^2 \right) . \tag{3.6}$$

This has the same form as the conifold; however important nontrivial dependences on the coordinate y have been introduced through the definitions of the vielbeins e^i.

As mentioned above, most of our intuition for constructing a deformation of (3.6) comes from the analogous case of the deformed conifold. Here we recall that the deformed conifold metric can be written as follows [17]:

$$\begin{aligned} ds^2_{DC} = \epsilon^{\frac{4}{3}} \frac{K(\tau)}{3} \Bigg(& \frac{1}{3K^3} \left[4\, d\tau^2 + (g^5)^2 \right] + \cosh(\tau)^2 \left[(g^3)^2 + (g^4)^2 \right] + \\ & + \sinh(\tau)^2 \left[(g^1)^2 + (g^2)^2 \right] \Bigg), \end{aligned} \tag{3.7}$$

where

$$K(\tau) = \frac{2^{2/3} (\sinh(4\tau) - 4\tau)^{1/3}}{2 \sinh(2\tau)} \,. \tag{3.8}$$

While the above is given in terms of a radial coordinate τ, a more natural coordinate for our analysis is given by r where $r^3 = \epsilon^2 \cosh(2\tau)$. Furthermore, to understand the first order deformation, is it also convenient to expand the metric for large r; more precisely, we expand in the dimensionless quantity ϵ^2/r^3. We find, up to second order in the deformation, that

$$\begin{aligned} ds^2_{DC(2)} = & \, dr^2 + r^2 \left(\frac{1}{6} \left[(g^1)^2 + (g^2)^2 + (g^3)^2 + (g^4)^2 \right] + \frac{1}{9} (g^5)^2 \right) + \\ & + \frac{1}{6} r^2 \left[-(g^1)^2 - (g^2)^2 + (g^3)^2 + (g^4)^2 \right] \frac{\epsilon^2}{r^3} + \\ & + \frac{1}{3} \left(-\frac{1}{6} r^2 [(g^3)^2 + (g^4)^2 + (g^1)^2 + (g^2)^2] + 2 \left[dr^2 + \frac{1}{9} (g^5)^2 r^2 \right] \right) \times \\ & \times \left(\ln \left(\frac{2r^3}{\epsilon^2} \right) - 1 \right) \frac{\epsilon^4}{r^6} + dr^2 \frac{\epsilon^4}{r^6} + \cdots . \end{aligned} \tag{3.9}$$

The zeroth order term is of course the undeformed conifold metric itself. The first order (in ϵ^2) term should be a solution to the linearized Einstein's equations, and so should be a zero mode of the Lichnerowicz operator (3.2). We make the simple observation that the first order contribution is transverse and traceless. The second order perturbation is "sourced" by the first order equations, and we have verified explicitly that the above system indeed satisfies the Ricci-flatness condition to second order.

We now turn our attention back to the cone over $Y^{p,q}$. We note that the combination $-(g^1)^2 - (g^2)^2 + (g^3)^2 + (g^4)^2$ showing up at first order in the above expansion is the same as $2(e^1e^3 + e^2e^4)$. This therefore suggests an Ansatz for a first order perturbation for the cone over $Y^{p,q}$ of the form

$$ds^2_{(1)} = h_{mn}dx^m dx^n = f(y)r^2\frac{\left(e^1e^3 + e^2e^4\right)}{3}\frac{\epsilon^2}{r^3}\,, \tag{3.10}$$

where we have included a function of y because of the non trivial dependence of the $Y^{p,q}$ metric on this coordinate. Inserting this Ansatz into the first order deformation equations, we find that $f(y)$ must satisfy a *first order ODE* for it to be a zero mode (even though the Lichnerowicz operator is *second* order). The solution is simply

$$f(y) = \frac{1}{(1-cy)^2}\,. \tag{3.11}$$

We have dropped an arbitrary multiplicative factor (it only becomes important in the case of a fully non-perturbative solution). Note that the first order perturbation is again transverse and traceless.

Despite some effort, we have unfortunately not been able to satisfy the second order deformation equations using the first order contributions as a source. It is likely that, at second order, the deformation of the metric will non-trivially depend on both r and y coordinates in an inherently non-separable manner. Although we have no direct proof, this belief is supported by various unsuccessful attempts at separating the functional dependence on r and y in the spirit of the deformed conifold metric (3.9).

When constructing the one-forms e^i, we have introduced angular ψ dependence by "rotating" e^ϕ and e^θ together using ψ. As mentioned above, one could instead have mixed e^y and e^β. This, however, does not alter the first order perturbation (up to irrelevant minus signs). Hence the other mixing (2.11), while it is perhaps distinct non-perturbatively, is indistinct to lowest order in ϵ^2. This might signal another possible type of deformation, but we will not explore it here.

As mentioned above, the rotation between e^ϕ and e^θ introduces dependence on the coordinate ψ. This dependence on ψ disappears in the zeroth order metric because e^ϕ and e^θ always appear in the SU(2) invariant combination. However this is no longer true for the first order perturbation. This dependence on ψ is dual to chiral symmetry breaking in the gauge theory, and will be discussed more explicitly in section 5.3.1 below.

3.2 Supersymmetry of the perturbed solution

Although the first order deformation was simply obtained by demanding Ricci-flatness to order ϵ^2, it turns out that it is in fact a complex deformation, at least to this order. To show this explicitly, we turn to the Killing spinor equation. In the absence of fluxes, the supersymmetry condition takes the form $\nabla\lambda = 0$, with the resulting λ a parallel spinor.

However, instead of working with the Killing spinor equation directly, we focus on the integrability condition

$$e^\mu_a e^\nu_b\,[\nabla_\mu, \nabla_\nu]\,\lambda = \frac{1}{4}R_{abcd}\Gamma^{cd}\lambda = 0\,. \tag{3.12}$$

Before obtaining the Riemann tensor in the tangent basis, we first make a convenient choice of vielbein. To do this, we note that we can absorb the first order term (3.10) into a shift of e^1 and e^2 according to

$$ds^2 = dr^2 + r^2\left(\frac{1}{6}\left[(e^1+\delta e^3)^2 + (e^2+\delta e^4)^2 + (e^3)^2 + (e^4)^2\right] + \frac{1}{9}(e^5)^2\right), \tag{3.13}$$

where we have defined the quantity

$$\delta = \frac{\epsilon^2}{r^3(1-cy)^2}\,. \tag{3.14}$$

This introduces a shift to second order in ϵ^2, which however is unimportant as we will always work only to first order in ϵ^2. We now make a natural choice of shifted vielbeins

$$\begin{aligned} \hat{e}^6_{(s)} &= dr\,, \quad & \hat{e}^1_{(s)} &= \frac{r}{\sqrt{6}}\left(e^1+\delta e^3\right), \quad & \hat{e}^2_{(s)} &= \frac{r}{\sqrt{6}}\left(e^2+\delta e^4\right), \\ \hat{e}^3_{(s)} &= \frac{r}{\sqrt{6}}e^3\,, \quad & \hat{e}^4_{(s)} &= \frac{r}{\sqrt{6}}e^4\,, \quad & \hat{e}^5_{(s)} &= \frac{r}{3}e^5\,. \end{aligned} \tag{3.15}$$

Using this shifted vielbein basis, we find that the integrability condition (3.12) is satisfied for spinors λ satisfying the simultaneous projections

$$\begin{aligned} \left(1+\Gamma^{1256}\right)\lambda &= 0\,, \\ \left(1-\Gamma^{1234}\right)\lambda &= 0\,. \end{aligned} \tag{3.16}$$

Writing out the $SO(6)$ generators in the spinor representation as $T^1 = \frac{i}{2}\Gamma^{12}$, $T^2 = \frac{i}{2}\Gamma^{34}$ and $T^3 = \frac{i}{2}\Gamma^{56}$, we see that parallel spinors have $SO(6)$ weights $(\frac{1}{2},-\frac{1}{2},\frac{1}{2})$ or $(-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$. Changing to an SU(4) basis with Cartan generators

$$H^1 = \frac{1}{2}(T^1+T^2), \qquad H^2 = \frac{1}{4\sqrt{3}}(-T^1+T^2+2T^3)\,, \qquad H^3 = \frac{1}{2\sqrt{6}}(T^1-T^2+T^3)\,, \tag{3.17}$$

we see that the parallel spinors are singlets of the SU(3) corresponding to H^1 and H^2. This is just the standard Calabi-Yau decomposition $\mathbf{4} \to \mathbf{3}+\mathbf{1}$ under SU(4) $\supset$ SU(3).

More explicitly, we have verified that the integrability condition (3.12) is satisfied at both zeroth and first order in ϵ^2 for the above projections. For example [1]

$$\begin{aligned} R_{53ab}\Gamma^{ab} &= \frac{6\epsilon^2}{r^5(1-cy)^2}\Gamma^{51}\left(1+\Gamma^{1256}\right), \\ R_{24ab}\Gamma^{ab} &= \frac{2(ac^2-1)}{r^2(1-cy)^3}\Gamma^{31}\left(1-\Gamma^{1234}\right) + \frac{3\sqrt{6}H(y)\epsilon^2 c}{r^5(1-cy)^3}\Gamma^{51}\left(1+\Gamma^{1256}\right), \end{aligned} \tag{3.18}$$

which are clearly zero when applied to λ.

[1] All components have been computed in Maple. Details of the calculations are available upon request to the authors.

4. Complex coordinates and Kähler potential

Having given the first order deformation and shown that it is supersymmetric, we now proceed to its classification. As a first step, we review the complex coordinates introduced in [3] with some minor adjustments to fit the present conventions [2]. The main new result of this section is an expression for the Kähler potential of the cone over $Y^{p,q}$.

We recall that ref. [3] obtained the complex coordinates for the cone over Sasaki-Einstein space. Let

$$\begin{aligned} \eta^1 &= \frac{1}{\sin\theta}d\theta - id\phi\,, \\ \tilde{\eta}^2 &= -\frac{dy}{H(y)^2} - i(d\beta - c\cos\theta d\phi)\,, \\ \tilde{\eta}^3 &= \frac{3dr}{r} + i[d\psi + \cos\theta d\phi + y(d\beta - c\cos\theta d\phi)]. \end{aligned} \tag{4.1}$$

In terms of these coordinates, one can write the metric as

$$ds^2 = r^2\frac{(1-cy)}{6}\sin^2\theta\eta^1\overline{\eta^1} + r^2\frac{H(y)^2}{6}\tilde{\eta}^2\overline{\tilde{\eta}^2} + \frac{r^2}{9}\tilde{\eta}^3\overline{\tilde{\eta}^3}\,. \tag{4.2}$$

Unfortunately, $\tilde{\eta}^2$ and $\tilde{\eta}^3$ are not integrable. However, integrable one-forms can be obtained by taking linear combinations of them:

$$\eta^2 = \tilde{\eta}^2 + c\cos\theta\eta^1\,, \qquad \eta^3 = \tilde{\eta}^3 + \cos\theta\eta^1 + y\tilde{\eta}^2\,. \tag{4.3}$$

In this case, $\eta^i = dz_i/z_i$ for $i = 1, 2, 3$, where

$$z_1 = \tan\frac{\theta}{2}e^{-i\phi}\,, \tag{4.4}$$

$$z_2 = \frac{(\sin\theta)^{2c}}{f_1(y)}e^{-i\beta}\,, \tag{4.5}$$

$$z_3 = r^3\frac{\sin\theta}{f_2(y)}e^{i\psi}\,, \tag{4.6}$$

with

$$f_1(y) = \exp\left(\int\frac{1}{H(y)^2}dy\right), \qquad f_2(y) = \exp\left(\int\frac{y}{H(y)^2}dy\right). \tag{4.7}$$

As a check, we may take the limit to the conifold (2.8), in which case the z_i's reduce to

$$z_1 \to \tan\frac{\theta_1}{2}e^{-i\phi}\,, \qquad z_2 \to \tan\frac{\theta_2}{2}e^{-i\beta}\,, \qquad z_3 \to r^3\sin\theta_1\sin\theta_2 e^{i\psi}\,, \tag{4.8}$$

along with $f_1 = \cot(\theta_2/2)$, $f_2 = \csc\theta_2$.

Returning to the Sasaki-Einstein case, note that (4.4) can be written as

$$\sin\theta = \frac{2\sqrt{z_1\bar{z}_1}}{1+z_1\bar{z}_1}\,, \tag{4.9}$$

[2] Note that we have taken $\phi \to -\phi$ with respect to [3].

where bars denote complex conjugation. Therefore

$$\frac{\partial \sin\theta}{\partial z_1} = \frac{1}{2z_1}\sin\theta\cos\theta\,, \qquad \frac{\partial \sin\theta}{\partial z_2} = 0\,, \qquad \frac{\partial \sin\theta}{\partial z_3} = 0\,. \tag{4.10}$$

From equation (4.5), an explicit expression for $f_1(y)$ can be obtained:

$$f_1(y)^2 = \frac{(\sin\theta)^{2c}}{z_2\bar{z}_2} = \sigma\,. \tag{4.11}$$

We are unable, however, to invert this equation to obtain an explicit expression of y in terms of σ. Nevertheless, the above equation suggests that if an explicit expression for y is possible, then the complex coordinates will enter into it only in the above combination denoted by σ. Moreover, y is independent of z_3 and its conjugate, i.e. $\partial y/\partial z_3 = \partial y/\partial \bar{z}_3 = 0$.

For suitable values of y, the total derivative of y can be evaluated:

$$\frac{dy}{d\sigma} = (\frac{d\sigma}{dy})^{-1} = \frac{wv}{12 f_1(y)^2}\,. \tag{4.12}$$

This is possible because y can be viewed as a function of σ only. In turn, σ is a function of z_1, z_2 and their conjugates. One obtains

$$\frac{\partial\sigma}{\partial z_1} = \frac{c\sigma}{z_1}\cos\theta\,. \tag{4.13}$$

Then

$$\frac{\partial y}{\partial z_1} = \frac{dy}{d\sigma}\frac{\partial\sigma}{\partial z_1} = \frac{1}{z_1}\frac{wv}{12}c\cos\theta\,. \tag{4.14}$$

Similarly, other partial derivatives can be evaluated. One can write all of them more succinctly using coordinates $u_i = \ln z_i$ as

$$\begin{aligned}
&\frac{\partial\theta}{\partial u_1} = \frac{1}{2}\sin\theta\,, && \frac{\partial\theta}{\partial u_2} = 0\,, && \frac{\partial\theta}{\partial u_3} = 0\,, && \frac{\partial\theta}{\partial \bar{u}_i} = \frac{\partial\theta}{\partial u_i}\,,\\
&\frac{\partial y}{\partial u_1} = \frac{wv}{12}c\cos\theta\,, && \frac{\partial y}{\partial u_2} = -\frac{wv}{12}\,, && \frac{\partial y}{\partial u_3} = 0\,, && \frac{\partial y}{\partial \bar{u}_i} = \frac{\partial y}{\partial u_i}\,,\\
&\frac{\partial r}{\partial u_1} = -\frac{r}{6}(1-cy)\cos\theta\,, && \frac{\partial r}{\partial u_2} = -\frac{ry}{6}\,, && \frac{\partial r}{\partial u_3} = \frac{r}{6}\,, && \frac{\partial r}{\partial \bar{u}_i} = \frac{\partial r}{\partial u_i}\,.
\end{aligned} \tag{4.15}$$

The metric can now be written in terms of the u_i as

$$\begin{aligned}
ds^2 = &\left[r^2\frac{(1-cy)}{6}\sin^2\theta + r^2\frac{wv}{36}c^2\cos^2\theta + \frac{r^2}{9}(1-cy)^2\cos^2\theta\right] du_1 d\bar{u}_1 + \\
&+\left[r^2\frac{wv}{36} + \frac{r^2}{9}y^2\right] du_2 d\bar{u}_2 + \frac{r^2}{9} du_3 d\bar{u}_3 + \\
&+\left[-r^2\frac{wv}{36}c\cos\theta + \frac{r^2}{9}y(1-cy)\cos\theta\right](du_1 d\bar{u}_2 + d\bar{u}_1 du_2) - \\
&-\frac{r^2}{9}y(du_2 d\bar{u}_3 + d\bar{u}_2 du_3) - \frac{r^2}{9}(1-cy)\cos\theta(du_1 d\bar{u}_3 + d\bar{u}_1 du_3)\,.
\end{aligned} \tag{4.16}$$

The coordinate r in the limit (2.8) reduces to the coordinate ρ as used by [17] for the conifold. Recall that the Kähler potential in the conifold case is ρ^2. Here, one can show that the Kähler potential for the Sasaki-Einstein cone is r^2. For example,

$$\begin{aligned}\partial_{u_1}\partial_{\bar{u}_1}r^2 &= \partial_{u_1}\left(-2r\frac{r}{6}(1-cy)\cos\theta\right)\\ &= -\left(\frac{1}{3}\frac{\partial r^2}{\partial u_1}(1-cy)\cos\theta+\frac{r^2}{3}\frac{\partial(-cy)}{\partial u_1}\cos\theta+\frac{r^2}{3}(1-cy)\frac{\partial\cos\theta}{\partial u_1}\right)\\ &= \frac{r^2}{9}(1-cy)^2\cos^2\theta+r^2\frac{wv}{36}c^2\cos^2\theta+\frac{r^2}{6}(1-cy)\sin^2\theta\,,\end{aligned}\tag{4.17}$$

which matches with the $(1,\bar{1})$ component of the metric written in (4.16). The other components of the metric can be similarly obtained. Thus the relation $g_{\mu\bar{\nu}}=\partial_\mu\partial_{\bar{\nu}}\mathcal{K}$ holds for the Kähler potential $\mathcal{K}=r^2$. This is an interesting result, and due to its simplicity it could provide alternative ways of understanding deformations and resolutions of the cone over $Y^{p,q}$. Similarly, the Kähler potential for the four-dimensional Kähler-Einstein base turns out to be $\frac{2}{3}\ln[(1+\frac{1}{z_1\bar{z}_1})f_2(y)]$.

4.1 First order perturbation in complex form

We now arrive at our goal of showing that the deformation obtained in the previous section is a complex deformation. Using the complex one-forms defined earlier, one obtains

$$\begin{aligned}-\sin\theta\eta^1\tilde{\eta}^2 &= \frac{1}{H(y)^2}dyd\theta+\sin\theta d\phi(d\beta-c\cos\theta d\phi)+\\ &\quad+i\left[d\theta(d\beta-c\cos\theta d\phi)-\frac{1}{H(y)^2}dy\sin\theta d\phi\right].\end{aligned}\tag{4.18}$$

In terms of these complex coordinates, the metric perturbation can be written as

$$\begin{aligned}2(e^1e^3+e^2e^4) &= 2\left[\cos\psi(e^\beta e^\phi+e^ye^\theta)+\sin\psi(e^\theta e^\beta-e^ye^\phi)\right]\\ &= -2H(y)\sqrt{1-cy}\;\left[\cos\psi\,\mathrm{Re}(-\sin\theta\eta^1\tilde{\eta}^2)+\sin\psi\,\mathrm{Im}(-\sin\theta\eta^1\tilde{\eta}^2)\right]\\ &= H(y)\sqrt{1-cy}\;\sin\theta\left[e^{i\psi}\eta^1\tilde{\eta}^2+e^{-i\psi}\bar{\eta}^1\bar{\tilde{\eta}}^2\right].\end{aligned}\tag{4.19}$$

The first order perturbation is of pure type. We note here that the $\eta^1\tilde{\eta}^2$ part is *by itself* a zero mode of the Lichnerowicz operator. Since the holomorphic (3,0)-form for the Sasaki-Einstein cone is known [3], a complex closed $h_{2,1}$ form can be constructed,

$$\begin{aligned}h_{(2,1)} = -\frac{1}{18}\frac{\epsilon^2}{(1-cy)^2}\Big[&H^2du^2\wedge du^3\wedge d\bar{u}^2+\sin^2\theta(1-cy)du^3\wedge du^1\wedge d\bar{u}^1+\\ &+cH^2\cos\theta du^3\wedge du^1\wedge d\bar{u}^2+\\ &+y(1-cy)\sin^2\theta du^1\wedge du^2\wedge d\bar{u}^1+\\ &+H^2\cos\theta du^1\wedge du^2\wedge d\bar{u}^2\Big].\end{aligned}\tag{4.20}$$

5. Warping the deformation

The general form of the solution we are seeking in this section has been explained in [8]. The main difference is that instead of warping the cone over $Y^{p,q}$ we need to warp the deformation of the cone. The solution contains a nontrivial F_5 representing the flux left by the D3 branes after the transition and G_3 which is the implication of considering fractional D3 branes, that is, D5 branes wrapping a 2-cycle inside the deformed cone. Thus the full IIB solution is given in terms of the fields

$$\begin{aligned} ds^2 &= h^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + h^{1/2}ds_6^2\,, \\ ds_6^2 &= dr^2 + r^2\left[(e^1_{(s)})^2 + (e^2_{(s)})^2 + (e^3_{(s)})^2 + (e^4_{(s)})^2 + (e^5_{(s)})^2\right], \\ F_5 &= (1+*)\mathcal{F}_5 = (1+*)dh^{-1}\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3\,, \\ G_3 &= -F_3 + \frac{i}{g_s}H_3 = iM\Omega_3\,. \end{aligned} \tag{5.1}$$

The shifted vielbein basis is given above in section 3.2. Given this general form of the solution, our goal is to find the explicit form of $h(r,y)$ and Ω_3. Furthermore, in the case when the deformation is zero, we expect to recover the solution of [8].

5.1 Turning on a G_3 flux in the deformed cone over $Y^{p,q}$

In this section we search for an appropriate complex G_3 which may be turned on in the deformed Sasaki-Einstein cone. To have a supersymmetric solution we would need G_3 to be a $(2,1)$ form [18]. However, we will not address the Dolbeault decomposition of G_3 directly. Rather, we will simply concentrate on finding a solution to the equations of motion with constant dilaton field. The constancy of the dilaton is readily satisfied by an imaginary self dual complex 3-form G_3. We thus consider G_3 to be proportional to an imaginary self dual 3-form Ω_3, namely

$$*_6\Omega_3 = i\Omega_3\,. \tag{5.2}$$

To assist in taking the Hodge dual while at the same time keeping the radial coordinate dependence explicit, we use a set of shifted vielbeins

$$\begin{aligned} &e^6_{(s)} = dr, \quad e^1_{(s)} = \left(e^1 + \delta e^3\right), \quad e^2_{(s)} = \left(e^2 + \delta e^4\right), \\ &e^3_{(s)} = e^3, \quad e^4_{(s)} = e^4\,, \quad e^5_{(s)} = e^5\,, \end{aligned} \tag{5.3}$$

which resemble the choice of (3.15), but with r-dependence (and some factors) removed.

In this shifted vielbein basis, the most general Ansatz for an imaginary self dual Ω_3 may be written as

$$\begin{aligned} \Omega_{(3)} = &\left(\frac{dr}{r} + i\frac{e^5}{3}\right)\wedge\Big(\alpha_1\left(e^1_{(s)}\wedge e^2_{(s)} + e^3_{(s)}\wedge e^4_{(s)}\right) + \alpha_2\left(e^1_{(s)}\wedge e^4_{(s)} + e^2_{(s)}\wedge e^3_{(s)}\right) + \\ &\qquad + i\alpha_3\left(e^1_{(s)}\wedge e^3_{(s)} - e^2_{(s)}\wedge e^4_{(s)}\right)\Big) \\ &+ \left(\frac{e^5}{3} + i\frac{dr}{r}\right)\wedge\Big(i\beta_1\left(e^1_{(s)}\wedge e^2_{(s)} - e^3_{(s)}\wedge e^4_{(s)}\right) + i\beta_2\left(e^1_{(s)}\wedge e^4_{(s)} - e^2_{(s)}\wedge e^3_{(s)}\right) + \end{aligned}$$

$$+\beta_3\left(e^1_{(s)}\wedge e^3_{(s)}+e^2_{(s)}\wedge e^4_{(s)}\right)\Bigg)+$$

$$+i\lambda_1\left(\frac{dr}{r}\wedge\frac{e^5_{(s)}}{3}\wedge e^1_{(s)}+\frac{i}{6}e^2_{(s)}\wedge e^3_{(s)}\wedge e^4_{(s)}\right)+$$

$$+\lambda_2\left(\frac{dr}{r}\wedge\frac{e^5_{(s)}}{3}\wedge e^2_{(s)}-\frac{i}{6}e^1_{(s)}\wedge e^3_{(s)}\wedge e^4_{(s)}\right)+$$

$$+i\lambda_3\left(\frac{dr}{r}\wedge\frac{e^5_{(s)}}{3}\wedge e^3_{(s)}+\frac{i}{6}e^1_{(s)}\wedge e^2_{(s)}\wedge e^4_{(s)}\right)+$$

$$+\lambda_4\left(\frac{dr}{r}\wedge\frac{e^5_{(s)}}{3}\wedge e^4_{(s)}-\frac{i}{6}e^1_{(s)}\wedge e^2_{(s)}\wedge e^3_{(s)}\right), \tag{5.4}$$

where the ten functions $\alpha_i, \beta_i, \lambda_i$ are general (possibly complex) functions. The factors of i in front of some of these functions are chosen for later convenience. We, however, restrict to the case when these functions depend only on r and y.

Our goal now is to expand the above equations in δ, that is in powers of ϵ^2/r^3, and then to require $\Omega_{(3)}$ to be closed. For this purpose we note that the algebra of the exterior derivative d acting on the basis e^i is

$$\begin{aligned}
de^1 &= -\left(\frac{y}{H}-K\right)e^1\wedge e^2+2Ke^3\wedge e^4\,,\\
de^2 &= 0\,,\\
de^3 &= -e^5\wedge e^4+Ke^2\wedge e^3-\frac{y}{H}e^1\wedge e^4\,,\\
de^4 &= e^5\wedge e^3+Ke^2\wedge e^4+\frac{y}{H}e^1\wedge e^3\,,\\
de^5 &= -e^1\wedge e^2+e^3\wedge e^4\,,
\end{aligned} \tag{5.5}$$

This follows directly from (2.7) and (2.9). We find that taking the exterior derivative of Ω_3 produces 14 out of 15 possible terms. To simplify matters we assume that we are looking for a solution that becomes that of [8] in the $\epsilon \to 0$ limit, and so we may take that all functions except α_1 are of order ϵ^2 already. This allows us to drop certain ϵ^4 terms. The actual computation is straightforward but quite involved, and we present some of the partial results in appendix B. The most important result is that such an imaginary self dual 3-form can be constructed explicitly, resulting in

$$\begin{aligned}
\Omega_3 = \left(\frac{dr}{r}+i\frac{e^5_{(s)}}{3}\right)\wedge\Bigg[&\frac{3}{2}\frac{1}{(1-cy)^2}(e^1_{(s)}\wedge e^2_{(s)}+e^3_{(s)}\wedge e^4_{(s)})+\\
&+\alpha_2(r,y)(e^1_{(s)}\wedge e^4_{(s)}+e^2_{(s)}\wedge e^3_{(s)})+\\
&+i\alpha_3(r,y)(e^1_{(s)}\wedge e^3_{(s)}-e^2_{(s)}\wedge e^4_{(s)})\Bigg]+\\
+b_1(r,y)\left(\frac{e^5_{(s)}}{3}+i\frac{dr}{r}\right)&\wedge\Big[(e^1_{(s)}\wedge e^3_{(s)}+e^2_{(s)}\wedge e^4_{(s)})+i(e^1_{(s)}\wedge e^4_{(s)}-e^2_{(s)}\wedge e^3_{(s)})\Big]+\\
+il_2(r,y)(e^3_{(s)}+ie^4_{(s)})&\wedge\left(\frac{dr}{r}\wedge\frac{e^5_{(s)}}{3}+\frac{1}{6}e^1_{(s)}\wedge e^2_{(s)}\right),
\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}
\alpha_2(r,y) &= \frac{3}{4}\frac{\epsilon^2}{r^3}\left(1-\frac{1}{(1-cy)^4}\right)+\\
&+\frac{3}{4}\frac{\epsilon^2}{r^3}\frac{1}{c^2\mathcal{Q}(y)}\left[(1-ac^2)\left(\frac{1}{(1-cy)^4}-1\right)-\frac{9}{(1-cy)^2}+\frac{11}{1-cy}+\right.\\
&\qquad\left.+4-6(1-cy)-3(1-cy)^2+3(1-cy)^3\right], \qquad (5.7)\\
\alpha_3(r,y) &= -\frac{3}{4}\frac{\epsilon^2}{r^3}\left(1-\frac{1}{(1-cy)^4}\right)+\\
&+\frac{3}{4}\frac{\epsilon^2}{r^3}\frac{1}{c^2\mathcal{Q}(y)}\left[(1-ac^2)\left(\frac{1}{(1-cy)^4}-1\right)-\frac{3}{(1-cy)^2}+\frac{5}{1-cy}-\right.\\
&\qquad\left.-8+6(1-cy)+3(1-cy)^2-3(1-cy)^3\right], \qquad (5.8)\\
b_1(r,y) &= \frac{3}{2}\frac{\epsilon^2}{r^3}\left[\ln\frac{2r^3}{\epsilon^2}-4+\frac{3}{(1-ac^2)(1-cy)}+\right.\\
&\qquad\left.+\frac{c}{2(1-ac^2)}\sum_{i=1}^{3}\frac{(a+2acy_i+(1-ac^2)y_i^2)}{y_i(1-cy_i)}\ln(|y-y_i|)\right],\\
l_2(r,y) &= \frac{9}{2}\frac{\epsilon^2}{r^3}\frac{1}{cH(1-cy)}\left[(1-cy)-\frac{1}{(1-cy)}\right]^2. \qquad (5.9)
\end{aligned}$$

Here $\mathcal{Q}(y)=a-3y^2+2cy^3$ and y_i are the three roots resulting from $\mathcal{Q}(y)=0$.

In the absence of a deformation ($\delta=\epsilon=0$) the above reduces to the simple result for $\Omega_{(3)}$ given by [8]

$$\alpha_1(y)=\frac{3}{2(1-cy)^2}\,. \qquad (5.10)$$

Moreover, in the limit of [8] $\Omega_{(3)}$ is a $(2,1)$ form, a fact that can be established using the complex coordinates of [3] as reviewed in section 4. Another interesting case, $c=0$ (when $K=0$) is the KS [7] solution with

$$\alpha_1\to\frac{3}{2}\,,\qquad \beta_2=\beta_3\to\frac{3}{2}\frac{\epsilon^2}{r^3}\left(\ln\left(\frac{2r^3}{\epsilon^2}\right)-1\right), \qquad (5.11)$$

and all other functions set to zero. For the KS solution the corresponding Ω_3 is also (2,1). In both of the above limiting cases, the fact that they are $(2,1)$ guarantees that the solution is supersymmetric. We have not addressed the question of the precise Dolbeault decomposition of our Ω_3; however we hope to return to this question in the future.

5.2 Other forms

By comparing the imaginary self dual 3-form obtained above with the Klebanov-Strassler solution [7] and the Herzog, Ejaz and Klebanov solution [8], we may extract the NS-NS 3-form flux H_3 and the R-R form F_3 from the real and imaginary parts of Ω_3. We find

$$\begin{aligned}
\frac{F_3}{M} &= \alpha_1(y)\left[e^1_{(s)}\wedge e^2_{(s)}\wedge\frac{e^5_{(s)}}{3}+e^3_{(s)}\wedge e^4_{(s)}\wedge\frac{e^5_{(s)}}{3}\right]+\\
&\quad +[\alpha_2(r,y)+b_1(r,y)]e^1_{(s)}\wedge e^4_{(s)}\wedge\frac{e^5_{(s)}}{3}+[\alpha_2(r,y)-b_1(r,y)]e^2_{(s)}\wedge e^3_{(s)}\wedge\frac{e^5_{(s)}}{3}+\\
&\quad +[\alpha_3(r,y)+b_1(r,y)]e^1_{(s)}\wedge e^3_{(s)}\wedge\frac{dr}{r}+[-\alpha_3(r,y)+b_1(r,y)]e^2_{(s)}\wedge e^4_{(s)}\wedge\frac{dr}{r}-\\
&\quad -l_2(r,y)e^3_{(s)}\wedge\frac{e^5_{(s)}}{3}\wedge\frac{dr}{r}+\frac{1}{6}l_2(r,y)e^1_{(s)}\wedge e^2_{(s)}\wedge e^3_{(s)}\,, \qquad (5.12)\\
\frac{H_3}{g_sM} &= \alpha_1(y)\left[e^1_{(s)}\wedge e^2_{(s)}\wedge\frac{dr}{r}+e^3_{(s)}\wedge e^4_{(s)}\wedge\frac{dr}{r}\right]+\\
&\quad +[\alpha_2(r,y)-b_1(r,y)]e^1_{(s)}\wedge e^4_{(s)}\wedge\frac{dr}{r}+[\alpha_2(r,y)+b_1(r,y)]e^2_{(s)}\wedge e^3_{(s)}\wedge\frac{dr}{r}+\\
&\quad +[-\alpha_3(r,y)+b_1(r,y)]e^1_{(s)}\wedge e^3_{(s)}\wedge\frac{e^5_{(s)}}{3}+[\alpha_3(r,y)+b_1(r,y)]e^2_{(s)}\wedge e^4_{(s)}\wedge\frac{e^5_{(s)}}{3}+\\
&\quad +l_2(r,y)e^4_{(s)}\wedge\frac{e^5_{(s)}}{3}\wedge\frac{dr}{r}-\frac{1}{6}l_2(r,y)e^1_{(s)}\wedge e^2_{(s)}\wedge e^4_{(s)}\,. \qquad (5.13)
\end{aligned}$$

The NS-NS 3-form flux can be derived from a potential $H_3 = dB_2$ where

$$\begin{aligned}
\frac{B_2}{g_sM} &= f(r,y)[e^1_{(s)}\wedge e^2_{(s)}+e^3_{(s)}\wedge e^4_{(s)}]+\\
&\quad +\left[-\frac{1}{3}\alpha_3(r,y)+\frac{1}{3}b_1(r,y)-\delta f(r,y)\right]e^1_{(s)}\wedge e^4_{(s)}+\\
&\quad +\left[-\frac{1}{3}\alpha_3(r,y)-\frac{1}{3}b_1(r,y)+\delta f(r,y)\right]e^2_{(s)}\wedge e^3_{(s)}+\frac{1}{3}l_2(r,y)e^3_{(s)}\wedge\frac{dr}{r}\,, \qquad (5.14)
\end{aligned}$$

with

$$f(r,y)=\frac{1}{2}\frac{1}{(1-cy)^2}\left[\ln\left(\frac{2r^3}{\epsilon^2}\right)-1\right]. \qquad (5.15)$$

If we take the limit (2.8), then only the term proportional to $f(r,y)$ survives. This is in agreement with the Klebanov-Strassler expression for B_2 expanded to first order in ϵ^2, which is

$$\frac{B_2}{g_sM}=\frac{1}{2}\left[\ln\left(\frac{2r^3}{\epsilon^2}\right)-1\right]\left[e^1_{(s)}\wedge e^2_{(s)}+e^3_{(s)}\wedge e^4_{(s)}\right]+\mathcal{O}(\epsilon^4). \qquad (5.16)$$

Due to self duality of the complex 3-form, the dilaton field is constant, $\phi=0$. Since $F_{3\,\mu\nu\rho}H_3{}^{\mu\nu\rho}=0$, the axion vanishes as well. The five-form flux is

$$\begin{aligned}
F_5 &= \mathcal{F}_5+*\mathcal{F}_5\,,\\
\mathcal{F}_5 &= B_2\wedge F_3\,,\\
\frac{1}{g_sM^2}\mathcal{F}_5 &= \frac{2}{3}f(r,y)\alpha_1(y)e^1_{(s)}\wedge e^2_{(s)}\wedge e^3_{(s)}\wedge e^4_{(s)}\wedge e^5_{(s)}-\\
&\quad -\frac{l_2(r,y)}{6(1-cy)^2}\ln\left(\frac{2r^3}{\epsilon^2}\right)e^1_{(s)}\wedge e^2_{(s)}\wedge e^3_{(s)}\wedge e^5_{(s)}\wedge\frac{dr}{r}+\mathcal{O}(\epsilon^4)\,. \qquad (5.17)
\end{aligned}$$

The effect of the inhomogenous metric is reflected in the five-form at first order in ϵ^2.

5.3 The warp factor

The equation for the warp factor can be extracted from the F_5 equation of motion

$$dF_5 = H_3 \wedge F_3 \,. \tag{5.18}$$

The form of F_5 from (5.1) implies that

$$d * dh^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = H_3 \wedge F_3 \,, \tag{5.19}$$

where

$$\begin{aligned} *\mathcal{F}_5 &= *dh^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= -r^5 h' \, e^1_{(s)} \wedge e^2_{(s)} \wedge e^3_{(s)} \wedge e^4_{(s)} \wedge e^5_{(s)} + r^3 \dot{h} H(y) \, e^r \wedge e^1_{(s)} \wedge e^3_{(s)} \wedge e^4_{(s)} \wedge e^5_{(s)} - \\ &\quad - r^3 \dot{h} H(y) \, \delta \, e^r \wedge e^1_{(s)} \wedge e^2_{(s)} \wedge e^3_{(s)} \wedge e^5_{(s)} \,, \end{aligned} \tag{5.20}$$

where in the last equation we used the fact that $dy = -H(y) \left(e^2_{(s)} - \delta \, e^4_{(s)} \right)$.

5.3.1 D3 branes on the deformation

We consider first the simple case where there are no fractional D3 branes, that is, the case of zero three-form flux. The above equations simplify schematically to the familiar

$$\nabla^2 h = \frac{1}{\sqrt{g}} \partial_m \left(\sqrt{g} g^{mn} \partial_n h \right) = \delta(\vec{x}) \,. \tag{5.21}$$

The first order perturbation is traceless, implying that the first order correction to $\det g$ is zero. Further, if we take that the D3 branes are at the tip of the cone, the zeroth order harmonic function is simply $A + B/r^4$. The metric perturbation has no r indices, and so the resulting expansion in ϵ^2 of the above equation trivially yields

$$\nabla^2 h_1 = 0 \,, \tag{5.22}$$

where h_1 is the ϵ^2 correction of h. The first order correction to the harmonic function can therefore be set to 0 consistently.

Given this fact, taking the usual limit $N \to \infty$, with $g_s N$ fixed gives the following geometry

$$ds^2 = R^2 \left[ds^2_{AdS_5} + ds^2_{Y^{p,q}} + \frac{\epsilon^2}{r^3} \left(\frac{\left(e^1 e^3 + e^2 e^4 \right)}{3(1 - cy)^2} \right) \right] . \tag{5.23}$$

As we mentioned before, this deformation of the solution corresponds to chiral symmetry breaking. To see this, note that the vector $\partial / \partial \psi$ is no longer a Killing vector for the perturbed metric. The new metric therefore also breaks the $U(1)_R$ symmetry (of the gauge theory) associated with (2.2) [4]

$$3 \frac{\partial}{\partial \psi} - \frac{1}{2} \frac{\partial}{\partial \alpha} \,, \tag{5.24}$$

and so chiral symmetry has been broken. Let us see this argument using standard AdS/CFT arguments. To a supergravity deformation of the form

$$a r^{\Delta-4} + b r^{-\Delta}\,, \tag{5.25}$$

corresponds a field theory deformation by an operator $\mathcal{O}$ such that

$$\mathcal{H} \to \mathcal{H} + a\mathcal{O}\,, \qquad \langle \mathcal{O} \rangle = b\,. \tag{5.26}$$

Applying this to our situation, we have a deformation that goes as

$$\epsilon^2\, r^{-3}\,, \tag{5.27}$$

which implies an expectation value for a dimension-3 operator which should be of the form

$$\langle \bar{\Psi}\Psi \rangle = \epsilon^2\,. \tag{5.28}$$

Thus the gauge theory living in the worldvolume of a stack of D3 branes at the tip of the complex deformation of the cone over $Y^{p,q}$ corresponds to placing the superconformal quiver gauge theory in a different vacuum where the operator $\bar{\Psi}\Psi$ has a nonzero vacuum expectation value.

5.3.2 D3 branes and fractional D3 branes

The problem of solving for the warp factor in the presence of fractional D3 branes on $B^{p,q}$ was addressed in [8]. We will take their solution as the zeroth order in epsilon term (as our solution collapses to theirs in the $\epsilon \to 0$ limit). Curiously, their solution remains unperturbed, as will be shown here. First, consider the resulting equation for h,

$$-\nabla^2_{(6)} h = \frac{1}{6}\,|H_3|^2\,. \tag{5.29}$$

Next, note that in the first order deformed solution none of the indices (in vielbein basis) on the $\mathcal{O}(\epsilon^2)$ terms agree with those of the $\mathcal{O}(\epsilon^0)$ term. The basis is diagonal, and so the zeroth order term with indices contracted with the first order term vanishes. Therefore, there are no $\mathcal{O}(\epsilon^2)$ source terms on the right hand side of the above equation. As before, the zeroth order solution takes care of the right hand side, leaving us with only the first order equation

$$\frac{1}{\sqrt{g}}\partial_m\left[\sqrt{g}g^{mn}\partial_n h_1(r,y)\right] = \frac{1}{\sqrt{g}}\partial_m\left[\sqrt{g}h^{mn}\partial_n h_0(r,y)\right], \tag{5.30}$$

where we read the right hand side of the equation as being a source term. Without the fractional D3 branes, the source had vanished because h_0 only depended on r, and h^{mn} has no r indices. However, now h_0 depends both on r and y. Interestingly, however, the right hand side still vanishes because

$$\sqrt{g}h^{my}\partial_y h_0(r,y) = \frac{\epsilon^2\sqrt{3}}{54\mathcal{Q}(y)^{1/2}}\begin{bmatrix} 0 \\ -\cos(\psi)\dot{h}_0(r,y)\sin(\theta) \\ -\sin(\psi)\dot{h}_0(r,y) \\ 0 \\ -c\cos(\theta)\sin(\psi)\dot{h}_0(r,y) \\ \cos(\theta)\sin(\psi)\dot{h}_0(r,y) \end{bmatrix}, \tag{5.31}$$

where this has been written in $[r, \theta, \phi, y, \beta, \psi]$ order. The divergence of this quantity obviously vanishes, with the first non-zero and last non-zero terms canceling after the appropriate derivatives are taken. Thus, again, we may consistently set the first order perturbation to the warp factor to be zero because the source term is absent.

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A. Gravitational perturbation theory

In section 3, we have relied heavily on linearized gravity. We simply note here the equations to second order. First, we write the metric perturbed to second order

$$\tilde{g}_{mn} = {}^0g_{mn} + {}^1h_{mn} + {}^2h_{mn}\,, \tag{A.1}$$

where the pre-superscripts denote the order of the perturbation. Given the above decomposition, the following expressions are valid to second order

$$\begin{aligned}
\tilde{g}^{ab} &= {}^0g^{ab} - {}^1h^{ab} - {}^2h^{ab} + {}^1h^a_c\,{}^1h^{cb}\,,\\
\tilde{\Gamma}^a_{bc} &= {}^0\Gamma^a_{bc} + {}^1\Gamma^a_{bc} + {}^2\Gamma^a_{bc} + {}^{(1,1)}\Gamma^a_{bc}\,,\\
\tilde{R}^a_{bcd} &= {}^0R^a_{bcd} + {}^1\Gamma^a_{bd;c} - {}^1\Gamma^a_{bc;d} + {}^2\Gamma^a_{bd;c} - {}^2\Gamma^a_{bc;d} + {}^{(1,1)}\Gamma^a_{bd;c} - {}^{(1,1)}\Gamma^a_{bc;d} + \\
&\quad + {}^1\Gamma^a_{ec}\,{}^1\Gamma^e_{bd} - {}^1\Gamma^a_{ed}\,{}^1\Gamma^e_{bc}\,.
\end{aligned} \tag{A.2}$$

We have defined the convenient quantities

$$\begin{aligned}
{}^1\Gamma^a_{bc} &= \frac{1}{2}\left({}^1h^a_{b;c} + {}^1h^a_{c;b} - {}^1h_{bc}{}^{;a}\right),\\
{}^2\Gamma^a_{bc} &= \frac{1}{2}\left({}^2h^a_{b;c} + {}^2h^a_{c;b} - {}^2h_{bc}{}^{;a}\right),\\
{}^{(1,1)}\Gamma^a_{bc} &= -\frac{1}{2}\,{}^1h^{ad}\left({}^1h_{db;c} + {}^1h_{dc;b} - {}^1h_{bc;d}\right).
\end{aligned} \tag{A.3}$$

In all of the above equations ${}^0g^{ab}$ is used to raise and lower indices, and to construct the Christoffel symbols used in the covariant derivatives. Constructing Einstein's equations to second order is now trivial. For the purposes of this paper, all first order perturbations are transverse, traceless and are zero modes of the Lichnerowicz operator. In addition, the background is Ricci flat. This greatly simplifies the linearized Einstein's equations, and they become

$$\begin{aligned}
\frac{1}{2}\Big[{}^2h^a_{b;d;a} + {}^2h^a_{d;b;a} - {}^2h_{bd;a}{}^{;a} - {}^2(h^a_a)_{;b;d}\Big] = \frac{1}{2}\Big[{}^1h^{ac}\left({}^1h_{cb;d;a} + {}^1h_{cd;b;a} - {}^1h_{bd;c;a} - {}^1h_{ac;b;d}\right)\Big] - \\
-\frac{1}{4}\,{}^1h^{ac}{}_{;d}\,{}^1h_{ac;b} - \\
-\frac{1}{2}\Big[{}^1h_{da;c}\,{}^1h_b{}^{ba;c} - {}^1h_{da;c}\,{}^1h_b{}^{bc;a}\Big].
\end{aligned} \tag{A.4}$$

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We may read this as the second order term being sourced by the first order term. In addition, we may perform a simple check of the above equation. In the case when ${}^1h$ is zero, the leading contribution to the metric is ${}^2h$. Therefore, what we have done should collapse to "leading order" perturbation theory. This is indeed the case, as we can see the Lichnerowicz operator acting on ${}^2h$ on the left hand side of the equation.

B. Explicit derivation of Ω_3

Since one of the main results supporting the existence of a chiral symmetry broken phase for the cascading quiver theory relies on the existence of the imaginary self dual 3-form G_3, we present the details of its derivation below. In particular, we strive to make clear the assumptions that go into solving the system.

Imposing

$$d\Omega_3 = 0\,, \tag{B.1}$$

where Ω_3 is given in (5.4) results in the following system of equations for the ten quantities $\alpha_i, \beta_i, \lambda_i$:

1. $r\alpha_1' + r\beta_1' + H\dot{\lambda}_1 - \lambda_1\left(\frac{y}{H} - K\right) = 0\,,$
2. $-r\alpha_3' - 3\delta\alpha_1 - 3\alpha_2 + r\beta_3' + 3\beta_2 + \frac{y}{H}\lambda_4 = 0\,,$
3. $\delta\left(r\alpha_1' - 3\alpha_1\right) + r\alpha_2' + 3\alpha_3 + r\beta_2' + 3\beta_3 - \frac{y}{H}\lambda_3 = 0\,,$
4. $-\delta\left(r\alpha_1' - 3\alpha_1\right) + r\alpha_2' + 3\alpha_3 - r\beta_2' - 3\beta_3 - H\dot{\lambda}_3 + K\lambda_3 = 0\,,$
5. $r\alpha_3' - 3\delta\alpha_1 + 3\alpha_2 + r\beta_3' + 3\beta_2 - H\dot{\lambda}_4 + K\lambda_4 = 0\,,$
6. $r\alpha_1' - r\beta_1' + 2K\lambda_1 = 0\,,$
7. $-H\dot{\alpha}_3 + 2\frac{y}{H}\alpha_3 - H\dot{\beta}_3 + \frac{1}{3}\lambda_3 - \frac{1}{6}r\lambda_4' = 0\,,$
8. $-H\dot{\alpha}_1\delta - 4K\delta\alpha_1 - H\dot{\alpha}_2 + 2\frac{y}{H}\alpha_2 + H\dot{\beta}_2 - \frac{1}{6}r\lambda_3' + \frac{1}{3}\lambda_4 = 0\,,$
9. $-H\dot{\alpha}_1 + 4K\alpha_1 - H\dot{\beta}_1 + \frac{1}{6}r\lambda_1' + \frac{1}{3}\lambda_2 = 0\,,$
10. $H\dot{\alpha}_3 - 2\frac{y}{H}\alpha_3 - H\dot{\beta}_3 - \frac{1}{2}\lambda_3 = 0\,,$
11. $-H\dot{\alpha}_1\delta - 4K\delta\alpha_1 - H\dot{\alpha}_2 + 2\frac{y}{H}\alpha_2 - H\dot{\beta}_2 + \frac{1}{2}\lambda_4 = 0\,,$
12. $-H\dot{\alpha}_1 + 4K\alpha_1 + H\dot{\beta}_1 = 0\,,$
13. $\frac{2}{3}\beta_1 - \frac{1}{6}\left[H\dot{\lambda}_2 - \lambda_2\left(\frac{y}{H} + K\right)\right] = 0\,,$
14. $\frac{1}{3}\lambda_1 + \frac{1}{6}r\lambda_2' = 0\,,$

where dot means derivative with respect to y and prime with respect to r. Note that we have already taken into account the expansion to leading order in ϵ^2. In particular, all quantities except α_1 are viewed as $\mathcal{O}(\epsilon^2)$ terms. This is why δ (which is of order ϵ^2) only survives in combination with α_1.

These equations may be simplified by defining the new variables

$$\begin{aligned} B_1 &= \frac{1}{2}(\beta_2+\beta_3)\,, & B_2 &= \frac{1}{2}(\beta_2-\beta_3)\,, \\ l_1 &= \frac{1}{2}(\lambda_3+\lambda_4)\,, & l_2 &= \frac{1}{2}(\lambda_3-\lambda_4)\,, \\ A_1 &= \frac{1}{2}(\alpha_2+\alpha_3)\,, & A_2 &= \frac{1}{2}(\alpha_2-\alpha_3)\,, \end{aligned} \tag{B.2}$$

and by taking appropriate linear combinations.

The result is a set of equations involving only β_1, λ_1, λ_2 and α_1:

1. $2r\beta_1' + H\dot{\lambda}_1 - \left(K + \frac{y}{H}\right)\lambda_1 = 0\,,$
2. $2r\alpha_1' + H\dot{\lambda}_1 + \left(3K - \frac{y}{H}\right)\lambda_1 = 0\,,$
3. $-12H\dot{\beta}_1 + r\lambda_1' + 2\lambda_2 = 0\,,$
4. $-H\dot{\alpha}_1 + 4K\alpha_1 + H\dot{\beta}_1 = 0\,,$
5. $4\beta_1 - H\dot{\lambda}_2 + \left(K + \frac{y}{H}\right)\lambda_2 = 0\,,$
6. $2\lambda_1 + r\lambda_2' = 0\,,$

a second set involving B_1, l_2, A_2 and α_1:

7. $2rB_1' + 6B_1 + H\dot{l}_2 - \left(K + \frac{y}{H}\right) l_2 = -\delta(r\alpha_1' - 6\alpha_1)\,,$
8. $2rA_2' - 6A_2 - H\dot{l}_2 + \dot{H} l_2 = 0\,,$
9. $12H\dot{B}_1 + l_2 - rl_2' = 0\,,$
10. $4H\dot{A}_2 - \frac{8y}{H}A_2 + \frac{5}{3}l_2 + \frac{1}{3}rl_2' = -16K\delta\alpha_1\,,$

and finally a third set involving only B_2, l_1, A_1 and α_1:

11. $2rA_1' + 6A_1 - H\dot{l}_1 + \dot{H} l_1 = 0\,,$
12. $-2rB_2' + 6B_2 - H\dot{l}_1 + \left(K + \frac{y}{H}\right) l_1 = \delta r\alpha_1'\,,$
13. $-12H\dot{B}_2 + l_1 + rl_1' = 0\,,$
14. $-4H\dot{A}_1 + \frac{8y}{H}A_1 + \frac{5}{3}l_1 - \frac{1}{3}rl_1' = 16K\delta\alpha_1\,.$

Furthermore, since α_1 only enters the second and third sets of equations in conjunction with δ, the $\mathcal{O}(\epsilon^2)$ part of α_1 decouples from those sets of equations (the zeroth order part acts as a source). Hence if one is given the zeroth order form for α_1, then these sets of equations decouple. A good guess can be made by looking at the limiting cases of the Klebanov-Strassler [7] and Herzog, Ejaz and Klebanov [8] solutions. Based on this, we take α_1 to be given by

$$\alpha_1 = \frac{3}{2}\frac{1}{(1-cy)^2}\,. \tag{B.3}$$

This choice of α_1, together with $\lambda_1 = \lambda_2 = \beta_1 = 0$ then satisfies the first set of equations without any correction at $\mathcal{O}(\epsilon^2)$. Next, taking the r-dependence of A_1 to be r^{-3} and setting $B_2 = l_1 = 0$ is consistent with third set of equations. Equation 14 is then solved to give

$$A_1 = \frac{3}{4}\frac{\epsilon^2}{r^3}\frac{1}{c^2\mathcal{Q}(y)}\left[\frac{1-ac^2}{(1-cy)^4} - \frac{6}{(1-cy)^2} + \frac{8}{1-cy} - 3 + ac^2\right], \tag{B.4}$$

where $\mathcal{Q}(y) = a - 3y^2 + 2cy^3$. The integration constant is fixed such that one gets the Klebanov-Strassler solution in the limit (2.8).

The remaining second set of equations present more of a challenge. To check their consistency, we make an Ansatz for r dependence as follows:

$$\begin{aligned} B_1 &= \frac{3\epsilon^2}{2r^3}\left[\mu(y)\ln\frac{r^3}{\epsilon^2} - \nu(y)\right], \\ l_2 &= \frac{H}{r^3}\left[\theta(y)\ln\frac{r^3}{\epsilon^2} - \psi(y)\right], \\ A_2 &= \frac{1}{H^2 r^3}\left[\rho(y)\ln\frac{r^3}{\epsilon^2} - \tau(y)\right]. \end{aligned} \tag{B.5}$$

The system now reduces to eight equations for six functions of y:

1. $9\epsilon^2\dot{\mu} + 2\theta = 0\,,$
2. $18\epsilon^2\dot{\nu} + 4\psi + 3\theta = 0\,,$
3. $H^2\dot{\theta} - 2y\theta = 0\,,$
4. $9\epsilon^2\mu + 2y\psi - H^2\dot{\psi} = 9\dfrac{\epsilon^2}{(1-cy)^4}\,,$
5. $12\rho + H^4\dot{\theta} = 0\,,$
6. $H^4\dot{\psi} + 12\tau + 6\rho = 0\,,$
7. $2\dot{\rho} - 4\dfrac{K}{H}\rho + \dfrac{1}{3}H^2\theta = 0\,,$
8. $-\dot{\tau} + \dfrac{2K}{H}\tau - \dfrac{1}{6}H^2\psi + \dfrac{1}{4}H^2\theta + 6KH\dfrac{\epsilon^2}{(1-cy)^4} = 0\,.$

However, we now see that equation 7 can be obtained from equations 3 and 5. and equation 8 can be obtained from equation 4 with the help of equations 6, 1 and 7. This leaves us with six equations for six functions. They can be solved to give

$$
\begin{aligned}
\theta = \rho &= 0\,, \\
\mu &= 1\,, \\
\psi &= -\frac{27}{2}\frac{\epsilon^2}{c\mathcal{Q}(y)}\left[(1-cy)-\frac{1}{(1-cy)}\right]^2, \\
\tau &= \frac{3}{4}H^2\epsilon^2\left[\frac{1}{(1-cy)^4}-1\right]+\frac{3\epsilon^2}{4}\frac{y}{c(1-cy)}\left[(1-cy)-\frac{1}{(1-cy)}\right]^2, \\
\nu &= -\frac{2}{9\epsilon^2}\int\psi dy \\
&= 4-\ln 2-\frac{3}{(1-ac^2)(1-cy)}- \\
&\quad -\frac{c}{2(1-ac^2)}\sum_{i=1}^{3}\frac{(a+2acy_i+(1-ac^2)y_i^2)}{y_i(1-cy_i)}\ln(y-y_i)\,,
\end{aligned}
\tag{B.6}
$$

where y_i are the three roots of the cubic equation $\mathcal{Q}(y) = 0$. The integration constants are chosen such that Klebanov-Strassler solution is obtained in the limit (2.8). The resulting imaginary self dual 3-form is given in section 5 in eqns. (5.6)–(5.9).

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